An example of Fourier–Mukai partners of minimal elliptic surfaces *

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Abstract

Let X and Y be smooth projective varieties over $\mathbb{C}$. We say that X and Y are *D-equivalent* (or, X is a *Fourier–Mukai partner* of Y) if their derived categories of bounded complexes of coherent sheaves are equivalent as triangulated categories. The aim of this short note is to find an example of mutually D-equivalent but not isomorphic relatively minimal elliptic surfaces.

1 Introduction

Let X be a smooth projective variety over $\mathbb{C}$. The derived category $D(X)$ of X is a triangulated category whose objects are bounded complexes of coherent sheaves on X. A *Fourier–Mukai (FM) transform* relating smooth projective varieties X and Y is an equivalence of triangulated categories $\Phi : D(X) \to D(Y)$. If there exists an FM transform relating X and Y, we call X an *FM partner* of Y. We also say that X and Y are *D-equivalent*. Moreover we say that X and Y are *K-equivalent* if there exist a smooth projective variety Z and birational morphisms $f : Z \to X$, $g : Z \to Y$ such that $f^*K_X \sim g^*K_Y$. It is conjectured by Kawamata (Conjecture 1.2 in [7]) that given birationally equivalent smooth projective varieties X and Y, they are D-equivalent if and only if they are K-equivalent. In this note, we construct a counterexample to his conjecture. More precisely, we have:

Main Theorem . *(i) Let p be a positive integer. Then there is a rational elliptic surface $S(p)$ such that $S(p)$ has a singular fiber of type ${}_pI_0$ and at least three non-multiple singular fibers of different Kodaira's types.*

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(ii) Let N be a positive integer and p a prime number such that $p > 6(N-1)+1$. Then there are rational elliptic surfaces T_i, $(1 \leq i \leq N)$ such that $T_i \not\cong T_j$ for $i \neq j$ and every T_i is an FM partner of $S(p)$. As a special case, $S = S(11)$ has an FM partner T such that $T \not\cong S$. These S and T are birational, D-equivalent but not K-equivalent.

Note that if X and Y are K-equivalent, they are isomorphic in codimension 1 (Lemma 4.2, [7]). In particular, if surfaces S and T are not isomorphic, they are not K-equivalent. Hence, in (ii), the statement for $p = 11$ follows from the one for arbitrary p.

Before ending Introduction, we give a few remarks to Main Theorem. For a smooth projective variety X, it is an interesting problem to find the set of isomorphic classes of FM partners of X. In connection with this problem, we have the following.

Theorem 1.1 (Theorem 1.1, [2] and Theorem 1.6, [7]). *Assume that X and Y are D-equivalent smooth projective surfaces but not isomorphic to each other. Then we know that one of the following holds.*

(i) X and Y are K3 surfaces.

(ii) X and Y are abelian surfaces.

(iii) X and Y are elliptic surfaces with the non-zero Kodaira dimension $\kappa(X) = \kappa(Y)$.

Using Theorem 1.1, we obtain the complete answer to the problem mentioned above in dimension 2 ([2], see also [7]). It is well-known that the cases (i) and (ii) in Theorem 1.1 really occur. More strongly, we have:

Theorem 1.2 ([8] and [6]). *Let N be a positive integer. Then there are K3 (respectively, abelian) surfaces T_i, $(1 \leq i \leq N)$ such that $T_i \not\cong T_j$ for $i \neq j$ and all T_i's are D-equivalent each other.*

Our Main Theorem means that the case (iii) in Theorem 1.1 really occurs, and a similar result to Theorem 1.2 is true for elliptic surfaces.

In contrast to Main Theorem and Theorem 1.2, it is predicted that given a smooth projective variety X, the set of isomorphic classes of FM partners of X is finite. Actually this is known for the 2-dimensional case ([2] and [7]).

Notation and conventions. All varieties are defined over $\mathbb{C}$ and "elliptic surface" always means "relatively minimal elliptic surface" in this note. For a set I, we denote by $|I|$ the cardinality of I.

2 The proof of Main Theorem

We need some standard notation and results before giving the proof. Let $\pi : S \to C$ be an elliptic surface. For an object E of $D(S)$, we define the fiber degree of E

$$d(E) = c_1(E) \cdot f,$$

where f is a general fiber of π. Let us denote by $\lambda_{S/C}$ the highest common factor of the fiber degrees of objects of $D(S)$. Equivalently, $\lambda_{S/C}$ is the smallest number d such that there is a holomorphic d-section of π. For integers $a > 0$ and i with i coprime to $a\lambda_{S/C}$, by [1] there exists a smooth, 2-dimensional component $J_S(a, i)$ of the moduli space of pure dimension one stable sheaves on S, the general point of which represents a rank a, degree i stable vector bundle supported on a smooth fiber of π. There is a natural morphism $J_S(a, i) \to C$, taking a point representing a sheaf supported on the fiber $\pi^{-1}(x)$ of S to the point x. This morphism is a minimal elliptic fibration ([1]). Put $J^i(S) := J_S(1, i)$. Obviously, $J^0(S) \cong J(S)$, the Jacobian surface associated to S, and $J^1(S) \cong S$.

Fix an elliptic surface with a section $\pi : B \to C$. Let $\eta = \operatorname{Spec} k$ be the generic point of C, where $k = k(C)$ is the function field of C, and let $\overline{k}$ be the algebraic closure of k. Put $\overline{\eta} = \operatorname{Spec} \overline{k}$. We define the *Weil–Chatelet group* $WC(B)$ by the Galois cohomology $H^1(G, B_\eta(\overline{k}))$. Here $G = \operatorname{Gal}(\overline{k}/k)$ and $B_\eta(\overline{k})$ is the group of points of the elliptic curve B_η defined over $\overline{k}$. Suppose that we are given a pair (S, φ), where S is an elliptic surface $S \to C$ and φ is an isomorphism $J(S) \to B$ over C, fixing their 0-sections. Then we have a morphism

$$B_\eta \times S_\eta \to J(S)_\eta \times S_\eta \to S_\eta.$$

Here the first morphism is induced by $\varphi^{-1} \times id_S$ and the second is given by translation. We obtain a principal homogeneous space S_η of B_η. Since this correspondence is invertible and the group $H^1(G, B_\eta(\overline{k}))$ classifies isomorphic classes of principal homogeneous spaces of B_η, we know that $WC(B)$ consists of all isomorphic classes of pairs (S, φ). Here two pairs (S, φ) and (S', φ') are *isomorphic* if there is an isomorphism $\alpha : S \to S'$ over C, such that $\varphi' \circ \alpha_* = \varphi$, where $\alpha_* : J(S) \to J(S')$ is the isomorphism induced by α (fixing 0-sections).

$$\begin{tikzcd} J(S) \arrow[r, "\alpha_*"] \arrow[d, "\varphi"'] & J(S') \arrow[d, "\varphi'"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

There is a short exact sequence (page 185, [4] or page 38, [3])

$$0 \to \text{Ш}(B) \to WC(B) \to \bigoplus_{t \in C} H_1(B_t, \mathbb{Q}/\mathbb{Z}) \to 0,$$

if B is not the product $C \times E$, where E is an elliptic curve. The group Ш(B) is called *Tate–Shafarevich group* and it is the subgroup of $WC(B)$ which consists of all isomorphic classes of pairs (S, φ) such that S does not have multiple fibers. For a rational surface B, it is known that Ш(B) is trivial (Example 1.5.12, [5]).

Now we are in position to prove Main Theorem.

Proof. (i) By the Persson's list [9], there is a rational elliptic surface $B \to C$ having a section and three singular fibers of type III^*, I_2, I_1 (there are many other choices for B). Fix a point $t_0 \in C$ such that B_{t_0} is smooth. Take an element $\xi = (\xi_t)$ of $WC(B) \cong \bigoplus_{t \in C} H_1(B_t, \mathbb{Q}/\mathbb{Z})$ such that ξ_{t_0} is of order p and $\xi_t = 0$ for other t. Then the surface $\pi : S(p) \to C$ corresponding to ξ is an elliptic surface with desired singular fibers. We can check that $S(p)$ is rational, for instance, by Proposition 1.3.23, [5].

(ii) Put $S = S(p)$. Because every (-1)-curve on S is a p-section of π, we know that $\lambda_{S/C} = p$. For $i \in \mathbb{Z}$, there is an isomorphism $\varphi_i : J(J^i(S)) \to B$ such that $(J^i(S), \varphi_i)$ corresponds to $i\xi \in WC(B)$ ([3], page 38). By Theorem 2.2, each $J^i(S)$ is mutually D-equivalent for $1 \le i < p$. We can also conclude that $J^i(S)$ is rational, since $\kappa(J^i(S)) = -\infty$ and the Euler numbers $e(J^i(S))$ and $e(S)$ coincide by Proposition 2.3, [2] (we can check the rationality also by using Proposition 1.3.23, [5]). Put

$$I = \{1, \dots, p-1\}, \qquad I(a) = \{i \in I \mid J^i(S) \cong J^a(S)\}$$

for $a \in I$. Then there are $i_1, \dots, i_M \in I$ such that $I = \coprod_{k=1}^{M} I(i_k)$ (disjoint union).

Claim 2.1. *For all $a \in I$, $|I(a)| \le 6$.*

If Claim 2.1 is true, we have $6M \ge |I| = p - 1$. By the assumption $p > 6(N-1)+1$, we have $M \ge N$, which completes the proof of Main Theorem.

Let us start the proof of Claim 2.1.

Step 1. For each $i \in I(a)$, we fix an isomorphism $\alpha_i : J^a(S) \to J^i(S)$. Because the rational surface $J^a(S)$ has a unique elliptic fibration, there exists $\delta \in \operatorname{Aut} C$ such that the following diagram is commutative.

$$\begin{array}{ccc} J^a(S) & \xrightarrow{\alpha_i} & J^i(S) \\ \downarrow & & \downarrow \\ C & \xrightarrow{\delta} & C \end{array}$$

This makes the following diagram commutative.

$$\begin{array}{ccc} J(J^a(S)) & \xrightarrow{\varphi_a^{-1}\circ\varphi_i\circ\alpha_{i*}} & J(J^a(S)) \\ \downarrow & & \downarrow \\ C & \xrightarrow{\delta} & C \end{array}$$

By our assumption, $J(J^a(S))$ has at least three singular fibers of different Kodaira's types. Hence δ must be the identity on $C \cong \mathbb{P}^1$ and then we can say that every α_i is an isomorphism over C.

Step 2. By Step 1, we know that $\varphi_i \circ \alpha_{i*} \circ \varphi_a^{-1}$ is an automorphism of B over C, fixing the 0-section. Put $\gamma_i = \varphi_i \circ \alpha_{i*} \circ \varphi_a^{-1}$.

$$\begin{array}{ccc} J(J^a(S)) & \xrightarrow{\alpha_{i*}} & J(J^i(S)) \\ {\scriptstyle\varphi_a}\downarrow & & \downarrow{\scriptstyle\varphi_i} \\ B & \xrightarrow{\gamma_i} & B \end{array}$$

Suppose $\gamma_i = \gamma_j$ for $i, j \in I(a)$, then by the isomorphism $\alpha_j \circ \alpha_i^{-1}$, we see that $(J^i(S), \varphi_i)$ is isomorphic to $(J^j(S), \varphi_j)$ and hence $i\xi = j\xi$ in $WC(B)$. Because the order of ξ is p, we obtain $i = j$. Since the order of the group of automorphism of B over C fixing the 0-section is at most 6, we get $|I(a)| \leq 6$. This finishes the proof. $\square$

Theorem 2.2 (Proposition 4.4, [2]). *Let $\pi : S \to C$ be an elliptic surface and T a smooth projective variety. Assume that the Kodaira dimension $\kappa(S)$ is non-zero. Then the following are equivalent.*

(i) T is an FM partner of S.

(ii) T is isomorphic to $J^b(S)$ for some integer b with $(b, \lambda_{S/C}) = 1$.

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